K25P 0953

Reg. No. : ....

Name : ....

IV Semester M.Sc. Degree (C.B.C.S.S. – OBE-Regular)
Examination, April 2025
(2023 Admission)
MATHEMATICS
MSMAT04C14 : Operator Theory

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any five** questions from this Part. **Each** question carries **4** marks. **(5×4=20)**

1. Let $T : X \to Y$ be a bounded linear operator, where X and Y are normed spaces and g be any bounded linear functional on Y. Then prove that the functional $f(x) = g(Tx)$, $x \in X$ is linear.

2. Define canonical mapping C from a normed space X into the second dual space X''. Prove that the canonical mapping C is an isimorphism of the normed space X onto the normed space R(C), the range of C.

3. Explain the uniform approximation on [a, b].

4. Describe the Gram determinant of $y_1, y_1, ..., y_n$.

5. Show that total boundedness implies boundedness, but the converse does not generally hold.

6. When can you say that a bounded self-adjoint linear operator is positive ? Prove that the sum of positive operators is positive.

P.T.O.

PART – B

Answer **any three** questions from this Part. **Each** question carries **7** marks. **(3×7=21)**

7. If a linear operator $T : R^n \to R^n$ is represented by a matrix T_E, then prove that the adjoint operator T^X is represented by the transpose of T_E.

8. Give an example of an operator with a spectral value but which is not an eigenvalue.

9. Let X and Y be normed spaces and $T : X \to Y$ a linear operator. Then prove that T is compact if and only if it maps every bounded sequence (x_n) in X onto a sequence (Tx_n) in Y which has convergent subsequence.

10. Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Then Prove that,
    a) All the eigenvalues of T are equal, and
    b) Eigenvectors corresponding to different eigenvalues of T are orthogonal.

11. Prove that the spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T : H \to H$ on a complex Hilbert space H lies in the closed interval [m, M] on the real axis, where $m = \inf_{\|x\|=1} \langle Tx, x \rangle$ and $M = \sup_{\|x\|=1} \langle Tx, x \rangle$.

PART – C

Answer **any three** questions from this Part. **Each** question carries **13** marks.

(3×13=39)

12. Define a closed linear operator from a normed space X to a normed space Y. State and prove the closed graph theorem.

13. Prove that the resolvent set $\rho(T)$ of a bounded linear operator T on a complex Banach space X is open.

14. Let $T : X \to Y$ be a linear operator. Prove that if T is compact, so is its adjoint operator $T^{X} : Y' \to X'$, where X and Y are normed spaces and X' and Y' are their dual spaces respectively.

15. Let $T : X \to X$ be a compact linear operator on a normed space X and let $\lambda \neq 0$. Then prove that there exists a smallest integer r (depending on λ) such that from n = r on, the null spaces $N(T_\lambda^n)$ are all equal, and if r > 0, the inclusions $N(T_\lambda^0) \subset N(T_\lambda^1) \subset \ldots \subset N(T_\lambda^r)$ are all proper.

16. Let (T_n) be a sequence of bounded self-adjoint linear operators on a complex Hilbert space H such that $T_1 \leqq T_2 \leqq \ldots \leqq T_n \leqq \ldots \leqq K$ where K is a bounded self-adjoint linear operator on H. Suppose that any T_j commutes with K and with every T_m. Then prove that (T_n) is strongly operator convergent and the limit operator T is linear, bounded and self-adjoint and satisfies $T \leqq K$.